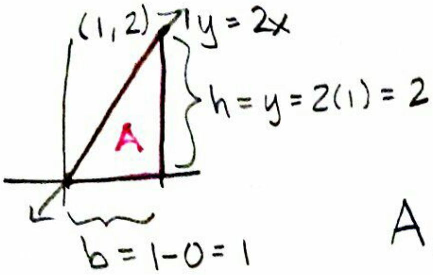
Lesson 29: Area and Riemann Sums

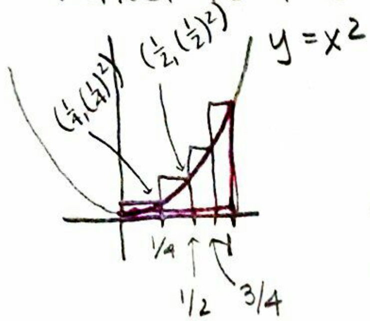
Ex 1 What is the area $\underbrace{\text{under } y=2x}_{\text{under } y=2x \text{ and above x-axis}}$ on $[0,1]$?

↑ x-values



$$A_\Delta = \tfrac{1}{2}bh = \tfrac{1}{2}(1)(2) = \boxed{1}$$

Ex 2 What is the area under $y=x^2$ on $[0,1]$?

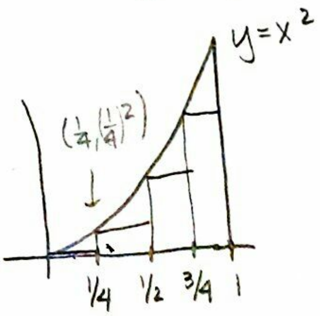


We'll approximate the area using 4 rectangles.

$$A \approx \tfrac{1}{4}\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4}(1)^2 \approx .469$$

This is a Right Riemann Sum, R_4

right ↑ ↑ # of rectangles



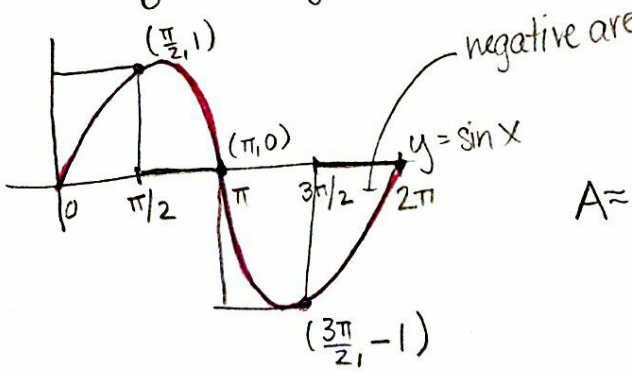
$$L_4 = \tfrac{1}{4}(0) + \tfrac{1}{4}\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^2 \approx .219$$

↑

Left Riemann Sum

Note: Height (and area) can be negative.

Ex 3 Approximate the (signed) area under $y=\sin x$ on $[0,2\pi]$ using a Right Riemann sum with 4 rectangles.



Length of $[0,2\pi] = 2\pi - 0 = 2\pi$

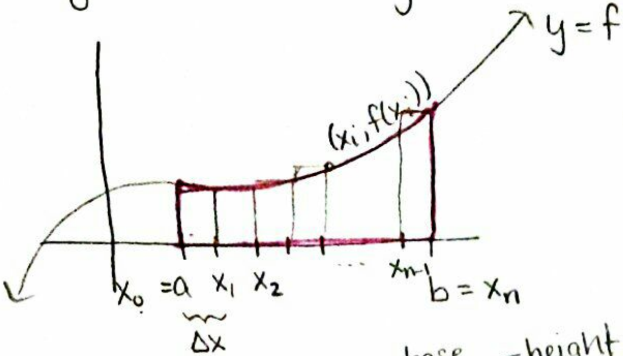
length of subintervals $= \frac{2\pi}{4} = \frac{\pi}{2}$

$$A \approx R_4 = \tfrac{\pi}{2}(1) + \tfrac{\pi}{2}(0) + \tfrac{\pi}{2}(-1) + \tfrac{\pi}{2}(0)$$
$$= \tfrac{\pi}{2} + 0 - \tfrac{\pi}{2} + 0 = \boxed{0}$$

Def $\sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n$, "the sum from $i=1$ to $i=n$ of a_i", is called sigma notation.

Ex 4 $(5-1)^2 + (6-1)^2 + (7-1)^2 = \sum_{i=5}^{7} (i-1)^2$

The Riemann sums estimating the area $y=f(x)$ on $[a,b]$ using n rectangles are:



$$\Delta x = \frac{b-a}{n}$$

$b-a$ ← length of interval

Δx ← length of subintervals

n ← # of rectangles

$$x_i = a + (\Delta x) i$$

$$\boxed{R_n = \sum_{i=1}^{n} \underbrace{\Delta x}_{\text{base}} \, \underbrace{f(x_i)}_{\text{height}} \qquad L_n = \sum_{i=0}^{n-1} \Delta x \, f(x_i)}$$

Ex 5 Estimate the area under $f(x) = \sqrt{x+7}$ on $[5,8]$ using 40 rectangles.

First, find $a, b, n, \Delta x, x_i$:

$a = 5, b = 8, n = 40, \Delta x = \frac{8-5}{40} = \frac{3}{40}, x_i = 5 + \frac{3}{40} i$

$$R_{40} = \sum_{i=1}^{40} \frac{3}{40} f\left(5 + \frac{3}{40} i\right) = \boxed{\sum_{i=1}^{40} \frac{3}{40} \sqrt{\left(5 + \frac{3}{40} i\right) + 7}} = \boxed{\sum_{i=1}^{40} \frac{3}{40} \sqrt{12 + \frac{3}{40} i}}$$

$$L_{40} = \boxed{\sum_{\underset{\uparrow}{i=0}}^{39} \underbrace{\frac{3}{40} \sqrt{12 + \frac{3}{40} i}}_{\text{same as } R_{40}}}$$

different bounds